ON THE CASTELNUOVO-MUMFORD REGULARITY
OF CONNECTED CURVES

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ABSTRACT. In this paper we prove that the regularity of a connected curve is
bounded by its degree minus its codimension plus 1. We also investigate the
structure of connected curves for which this bound is optimal. In particular,
we construct connected curves of arbitrarily high degree in $\mathbb{P}^4$ having maximal
regularity, but no extremal secants. We also show that any connected curve
in $\mathbb{P}^3$ of degree at least 5 with maximal regularity and no linear components
has an extremal secant.

0. INTRODUCTION

Let $S = \mathbb{K}[x_0, ..., x_n]$ where $\mathbb{K}$ is an algebraically closed field, and let M be a
graded module over S. We say that M is r-regular if the i-th syzygy module of M
is generated in degrees less than or equal to $r + i$ for all i. The regularity of M,
denoted $\operatorname{reg}(M)$, is defined to be the infimum of all r's such that M is r-regular. If
X is a subscheme of $\mathbb{P}^n$, then we define the regularity of X, denoted $\operatorname{reg}(X)$, to be
the regularity of the saturated homogeneous ideal of X, I_X.

In [4], Eisenbud and Goto conjectured that if X is a reduced nondegenerate
connected in codimension 1 subscheme of $\mathbb{P}^n$, then the regularity of X is at most
the degree of X minus the codimension of X plus 1. Various weakenings of this
conjecture are known to be true in low dimension, (see, for example, [6], [11], [10],
[12], [7], [8], [9]), but the general conjecture is still open. In this paper we prove
that this conjecture is true when X is a curve. We denote by $\operatorname{Span}(C)$ the linear
span of the curve C.

Main Theorem. *If C is a connected reduced curve in $\mathbb{P}^n$, then*

$$(1) \qquad \operatorname{reg}(C) \leq \deg(C) - \dim(\operatorname{Span}(C)) + 2.$$

In [6], Gruson, Lazarsfeld, and Peskine proved this result in the case when C
is assumed to be irreducible. We prove our theorem by using their theorem as a
base case and inducting on the number of irreducible components of C. The main
lemma that allows this to work is a result of G. Caviglia [1].

We also study the connected curves for which the Eisenbud-Goto bound is opti-
mal. In [6], Gruson, Lazarsfeld, and Peskine give a complete classification of such
curves in the irreducible case. They prove that, except for a few low degree excep-
tions, all such curves are smooth, rational, and have a secant line of degree equal

to the regularity of the curve. The existence of such a secant guarantees that the ideal of the curve will require a generator of degree equal to the regularity of the curve, so, at least for these curves, the regularity has to appear at the first step in a resolution for the ideal.

In the connected case, it turns out that there are curves of maximal regularity which have arbitrarily high degree in $\mathbb{P}^n$, $n \geq 4$ and which do not have such a secant. A picture of such a curve of degree 9 in $\mathbb{P}^4$ follows. (The labels of the various parts of the picture will be explained in Section 3.)
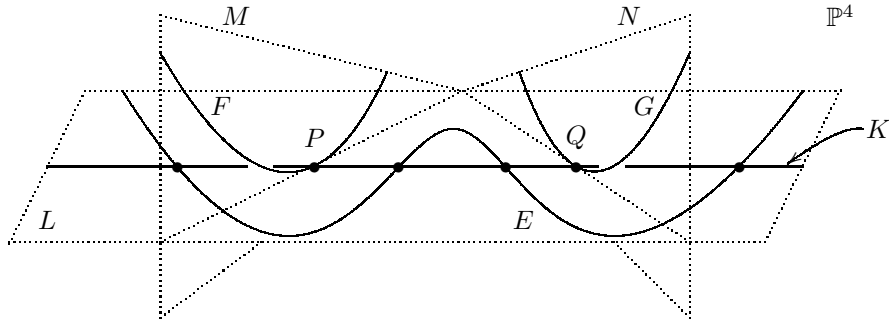
The ideal of this curve, however, requires a generator of degree equal to the regularity, and we are still unsure as to whether the ideal of any curve of maximal regularity, except for a few low degree exceptions, requires a generator of degree equal to the regularity of the curve.

In the final section we prove some structure theorems about connected curves of maximal regularity. The first result that we prove here is that the irreducible components of such a curve have to be fairly well spaced apart. To be precise, if C is a connected curve that is the union of two connected curves D and E having no common components, then we are able to show that $\dim(\mathrm{Span}(D) \cap \mathrm{Span}(E)) \leq 2$. Using this fact, we split our analysis of these curves into three cases depending on whether $\dim(\mathrm{Span}(D) \cap \mathrm{Span}(E))$ equals 0, 1, or 2. As a result of this analysis we are able to prove that any connected curve of maximal regularity in $\mathbb{P}^3$ of degree ≥ 5 having no linear components has an extremal secant. We are also able to show that any connected subcurve of a connected curve of maximal regularity is either a line or has maximal regularity.

1. Notation and conventions

Throughout this paper we use the following conventions:
- $\mathbb{K}$ is an algebraically closed field of arbitrary characteristic.
- $S = \mathbb{K}[x_0, ..., x_n]$, $\mathfrak{m} = (x_0, ..., x_n)$, and $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{K}} = \mathrm{Proj}(S)$.
- All ideals and modules over S are assumed to be graded.
- By a *curve* we mean a reduced one-dimensional scheme embedded in $\mathbb{P}^n$ for some n.

2. Proof of the Main Theorem

To prove the Main Theorem, we take any reducible connected curve and write it as the union of two nonempty connected curves, C and D, which meet each other in a finite set of points. By induction on the number of irreducible components in the curves, we may assume that C and D satisfy the inequality (1), so we need

only show that this implies that $C \cup D$ does. We will break the proof that $C \cup D$ satisfies (1) into three cases depending on the dimension of $\mathrm{Span}(C) \cap \mathrm{Span}(D)$. The following theorem of G. Caviglia allows us to bound the regularities of certain sums and intersections of ideals:

Theorem 2.1. *If $I, J \subset S$ are ideals and $\dim(S/(I + J)) \leq 1$, then $\mathrm{reg}(I + J) \leq \mathrm{reg}(I) + \mathrm{reg}(J) - 1$, and $\mathrm{reg}(I \cap J) \leq \mathrm{reg}(I) + \mathrm{reg}(J)$.*

Proof. See Corollary 3.4 in [1]. □

In order to simplify some of the statements that follow, we define $\Xi(C)$ to be $\deg(C) - \dim(\mathrm{Span}(C)) + 2$ for any curve C. With this definition the inequality (1) of the Main Theorem becomes

$$(2) \qquad \mathrm{reg}(C) \leq \Xi(C).$$

The above theorem tells us, among other things, how the regularity of a union of two curves depends on the regularity of each of the component curves. The following lemma, whose proof we leave to the reader, does the same thing for $\Xi(C)$.

Lemma 2.2. *If C and D are curves with no common component, then $\Xi(C \cup D) = \Xi(C) + \Xi(D) + \dim(\mathrm{Span}(C) \cap \mathrm{Span}(D)) - 2$ where we take the dimension of the empty scheme to be -1.* □

Combining Theorem 2.1 and Lemma 2.2 we obtain the following corollary, which establishes the inequality (2) for $C \cup D$ assuming that $\dim(\mathrm{Span}(C) \cap \mathrm{Span}(D)) \geq 2$.

Corollary 2.3. *If C and D are curves with no common component both satisfying equation (2) and $\dim(\mathrm{Span}(C) \cap \mathrm{Span}(D)) \geq 2$, then $C \cup D$ satisfies equation (2).*

Secondly we deal with the case when $\dim(\mathrm{Span}(C) \cap \mathrm{Span}(D)) = 1$. To do this we need the following lemmas.

Lemma 2.4. *If X and Y are finite subschemes of $\mathbb{P}^n$, then $\deg(X \cup Y) = \deg(X) + \deg(Y) - \deg(X \cap Y)$.*

Proof. There is an exact sequence

$$0 \rightarrow S/(I_X \cap I_Y) \rightarrow S/I_X \oplus S/I_Y \rightarrow S/(I_X + I_Y) \rightarrow 0.$$

The result follows as $S/(I_X + I_Y)$ is isomorphic to $S_{X \cap Y}$ in high degrees. □

Lemma 2.5. *If C is a curve satisfying (2), then $\Xi(C) \geq 2$.*

Proof. If C is a line, then

$$\Xi(C) = \deg(C) - \dim(\mathrm{Span}(C)) + 2 = 1 - 1 + 2 = 2.$$

If C is not a line, then $2 \leq \mathrm{reg}(C) \leq \Xi(C)$. □

Definition 2.6. We define the *saturation degree*, denoted $\mathrm{sat}(I)$, of an ideal $I \subset S$ to be $\mathrm{reg}(I^{sat}/I) + 1$. Equivalently it is the infimum of all d for which $I_e = I_e^{sat}$ for all $e \geq d$.

Notice that for any ideal $I \subset S$,

$$(3) \qquad \mathrm{reg}(I) = \max\{\mathrm{reg}(I^{sat}), \mathrm{sat}(I)\}.$$

This follows from the long exact sequence in local cohomology coming from the short exact sequence

$$0 \rightarrow I \rightarrow I^{sat} \rightarrow I^{sat}/I \rightarrow 0$$

after observing that $\mathrm{H}^0_{\mathfrak{m}}(I^{sat}) = \mathrm{H}^1_{\mathfrak{m}}(I^{sat}) = \mathrm{H}^i_{\mathfrak{m}}(I^{sat}/I) = 0$ for all $i \geq 1$. (We have defined regularity in terms of free resolutions, but it can also be characterized in terms of the vanishing of local cohomology. See chapter 4 of [3] for more details.) Also notice that if I and J are ideals of S, then

$$(4) \qquad\qquad \mathrm{sat}(I \cap J) \leq \max\{\mathrm{sat}(I), \mathrm{sat}(J)\}$$

as $(I \cap J)^{sat} = I^{sat} \cap J^{sat}$.

We can now prove the following proposition which establishes the inequality (2) for $C \cup D$ assuming that $\dim(\mathrm{Span}(C) \cap \mathrm{Span}(D)) = 1$.

Proposition 2.7. *If C and D are intersecting curves with no common component both satisfying* (2) *and* $\dim(\mathrm{Span}(C) \cap \mathrm{Span}(D)) = 1$, *then $C \cup D$ satisfies* (2).

Proof. Let $L_1 = \mathrm{Span}(C)$, $L_2 = \mathrm{Span}(D)$, and $L = \mathrm{Span}(C) \cap \mathrm{Span}(D)$. We separate the argument into two cases:

Case a: L is not a component of either C or D.

Let $J_1 = I_C + I_L$ and $J_2 = I_D + I_L$. Notice that

$$J_1 + J_2 = I_C + I_D + I_L = (I_C + I_{L_1}) + (I_D + I_{L_2}) = I_C + I_D$$

as $I_{L_1} \subset I_C$ and $I_{L_2} \subset I_D$. Therefore, from the exact sequence

$$0 \to J_1 \cap J_2 \to J_1 \oplus J_2 \to J_1 + J_2 = I_C + I_D \to 0$$

and Corollary 20.19 of [2] we deduce that

$$\mathrm{reg}(I_C + I_D) \leq \max\{\mathrm{reg}(J_1 \cap J_2) - 1, \mathrm{reg}(J_1), \mathrm{reg}(J_2)\}.$$

By equation (3) and inequality (4), we have

$$\mathrm{reg}(J_1 \cap J_2) \leq \max\{\mathrm{reg}((J_1 \cap J_2)^{sat}), \mathrm{reg}(J_1), \mathrm{reg}(J_2)\}.$$

By Theorem 2.1, $\mathrm{reg}(J_1) \leq \mathrm{reg}(C)$ and $\mathrm{reg}(J_2) \leq \mathrm{reg}(D)$. Therefore,

$$(5) \qquad \mathrm{reg}(I_C + I_D) \leq \max\{\mathrm{reg}((J_1 \cap J_2)^{sat}) - 1, \mathrm{reg}(C), \mathrm{reg}(D)\}.$$

The ideal of $(C \cap L) \cup (D \cap L)$ is $(J_1 \cap J_2)^{sat}$. Since a finite scheme on a line has regularity equal to its degree,

$$\mathrm{reg}((J_1 \cap J_2)^{sat}) = \deg((C \cap L) \cup (D \cap L)).$$

By Lemma 2.4,

$$\deg((C \cap L) \cup (D \cap L)) \leq \deg(C \cap L) + \deg(D \cap L) - \deg(C \cap D \cap L).$$

Since the nonempty intersection of C and D is contained in L,

$$\deg(C \cap D \cap L) \geq 1.$$

Since a scheme cannot have a secant of higher degree than its regularity, $\deg(C \cap L) \leq \mathrm{reg}(C)$ and $\deg(D \cap L) \leq \mathrm{reg}(D)$. Therefore,

$$\mathrm{reg}((J_1 \cap J_2)^{sat}) \leq \mathrm{reg}(C) + \mathrm{reg}(D) - 1.$$

Combining this with equation (5) it follows that

$$(6) \qquad\qquad \mathrm{reg}(I_C + I_D) \leq \mathrm{reg}(C) + \mathrm{reg}(D) - 2.$$

From equation (6) and the exact sequence

$$0 \to I_{C \cup D} \to I_C \oplus I_D \to I_C + I_D \to 0$$

it follows that $\mathrm{reg}(C \cup D) \leq \mathrm{reg}(C) + \mathrm{reg}(D) - 1$. Because C and D both satisfy (2), $\mathrm{reg}(C) + \mathrm{reg}(D) - 1 \leq \Xi(C) + \Xi(D) - 1 = \Xi(C \cup D)$ by Lemma 2.2. Therefore $C \cup D$ satisfies (2).

Case b: L is a component of one of C or D.

Without loss of generality, we assume that L is a component of C. In this case $I_C \subset I_L$ and therefore $I_C + I_D = I_L + I_D$ as $I_L \subset I_C + I_D$. Therefore

$$\mathrm{reg}(I_C + I_D) = \mathrm{reg}(I_L + I_D) \leq \mathrm{reg}(D)$$

by Theorem 2.1. From this and the exact sequence

$$0 \to I_{C \cup D} \to I_C \oplus I_D \to I_C + I_D \to 0$$

we deduce that $\mathrm{reg}(C \cup D) \leq \max\{\mathrm{reg}(C), \mathrm{reg}(D) + 1\}$. Because C and D both satisfy (2), this implies that $\mathrm{reg}(C \cup D) \leq \max\{\Xi(C), \Xi(D) + 1\}$. We conclude that $C \cup D$ satisfies (2) by Lemma 2.5 and Lemma 2.2. $\qquad\square$

The final case we need to deal with to complete the proof of the main theorem is the case when $\dim(\mathrm{Span}(C) \cap \mathrm{Span}(D)) = 0$. (Notice that we cannot have $\dim(\mathrm{Span}(C) \cap \mathrm{Span}(D)) = -1$ as C and D must intersect.) We do this in the following proposition.

Proposition 2.8. *If C and D are intersecting curves with no common component both satisfying* (2) *and* $\dim(\mathrm{Span}(C) \cap \mathrm{Span}(D)) = 0$, *then $C \cup D$ satisfies* (2).

Proof. Let $L_1 = \mathrm{Span}(C)$, $L_2 = \mathrm{Span}(D)$, and $P = \mathrm{Span}(C) \cap \mathrm{Span}(D)$. As in Case a of Proposition 2.7, $I_P = I_{L_1} + I_{L_2} \subset I_C + I_D \subset I_{C \cap D}$. However, $C \cap D$ is nonempty, so its ideal cannot strictly contain a prime of dimension 1. Therefore $I_P = I_C + I_D$. From the exact sequence

$$0 \to I_{C \cup D} \to I_C \oplus I_D \to I_C + I_D = I_P \to 0$$

and the fact that $\mathrm{reg}(I_P) = 1$, we see that

$$\mathrm{reg}(C \cup D) \leq \max\{\mathrm{reg}(C), \mathrm{reg}(D), 2\}.$$

Because C and D both satisfy (2), this implies that

$$\mathrm{reg}(C \cup D) \leq \max\{\Xi(C), \Xi(D), 2\}.$$

We conclude that $C \cup D$ satisfies (2) by Lemma 2.5 and Lemma 2.2. $\qquad\square$

3. Curves of maximal regularity: Introduction and examples

In this section we partially classify those curves for which the Eisenbud-Goto bound is optimal. Such a curve is said to have *maximal regularity*. The irreducible case is handled by Theorems 2.1 and 3.1 of [6]. Their result is stated in terms of extremal secant lines which we define here.

Definition 3.1. A linear subspace $L \subset \mathbb{P}^n$ is said to be an *extremal secant* of a curve $C \subset \mathbb{P}^n$ if $C \cap L$ is finite and $\mathrm{reg}(C \cap L) = \Xi(C)$.

By Corollary 3.10 of [5], if a connected curve C has an extremal secant of dimension i, then S/I_C has a minimal i-th syzygy of degree $\Xi(C) + i$. Hence, only curves of maximal regularity can have extremal secants. What Gruson, Lazarsfeld, and Peskine showed is that essentially all irreducible curves of maximal regularity have extremal secant lines, and it follows from their classification that all irreducible curves of maximal regularity have extremal secant hyperplanes. Specifically, they proved the following theorem.

Theorem 3.2. *An irreducible curve C has maximal regularity if and only if either $\Xi(C) = 3$ or C has an extremal secant line.*

We will prove that all connected curves which either have an extremal secant line or satisfy $\Xi(C) = 3$ have maximal regularity, but, unlike in the irreducible case, we will show how to construct connected curves of maximal regularity in $\mathbb{P}^n$ with no extremal secant line for any value of $\Xi(C) \geq 4$ and any $n \geq 4$. It would be interesting to know whether any such curves exist in $\mathbb{P}^3$. As we will see below, the construction we give necessitates $n \geq 4$.

We have already shown that if a connected curve C has an extremal secant line, then it has maximal regularity. We prove in Proposition 3.4 that all connected curves of regularity at most 2 have $\Xi(C) = 2$, so a connected curve with $\Xi(C) = 3$ must have regularity at least 3, hence exactly 3. Before proving this we need to define the following term which is used in the statement of the proposition.

Definition 3.3. We inductively define a *linearly normal tree of rational curves*, shortened to *tree* in the sequel. A linearly normal tree of rational curves with one component is defined to be a rational normal curve. A linearly normal tree of rational curves with k components, $k \geq 2$, is defined to be a curve C in $\mathbb{P}^n$ such that $C = C_0 \cup D$ where C_0 is a linearly normal tree of rational curves with $k - 1$ components, D is a rational normal curve and $C \cap D = \mathrm{Span}(C) \cap \mathrm{Span}(D)$ is a single point.

Proposition 3.4. *If C is a connected curve in $\mathbb{P}^n$, then the following statements are equivalent:*

> **(i)** *C is a linearly normal tree of rational curves.*
> **(ii)** *$\Xi(C) = 2$.*
> **(iii)** *$\mathrm{reg}(C) \leq 2$.*

Proof. The equivalence of conditions **(i)** and **(ii)** follows directly from Xambó's classification of connected in codimension 1 algebraic sets of minimal degree in [13]. The fact that **(ii)** implies **(iii)** follows directly from the main theorem, so we need only prove that **(iii)** implies **(ii)**.

If $\mathrm{reg}(C) = 1$, then C is a line which satisfies $\Xi(C) = 2$, so assume that $\mathrm{reg}(C) = 2$. Without loss of generality, we may assume that C is nondegenerate in $\mathbb{P}^n$. (Otherwise, replace $\mathbb{P}^n$ with $\mathrm{Span}(C)$.) From the exact sequence

$$0 \to \mathcal{I}_C \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_C \to 0$$

we see that the map on global sections $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}(n)) \to \mathrm{H}^0(\mathcal{O}_C(n))$ is surjective for all $n \geq 1$. Since C is connected it is surjective when $n = 0$. Therefore C is projectively normal which implies C is arithmetically Cohen-Macaulay as C is a curve. If $Z = H \cap C$ is a generic hyperplane section, then

$$\mathrm{reg}(I_{Z/H}) = \mathrm{reg}(I_Z) = \mathrm{reg}(I_C) = 2.$$

From the exact sequence

$$0 \to \mathcal{I}_{Z/H} \to \mathcal{O}_H \to \mathcal{O}_Z \to 0$$

we see that the map on global sections

$$\mathbb{K}^{n-1+1} \cong \mathrm{H}^0(\mathcal{O}_H(1)) \to \mathrm{H}^0(\mathcal{O}_Z(1)) \cong \mathbb{K}^{\deg(Z)} = \mathbb{K}^{\deg(C)}$$

is surjective. Therefore $n \geq \deg(C)$. Since C is nondegenerate, $\deg(C) \geq n$ by Lemma 2.5. Therefore $\Xi(C) = 2$. $\square$

Corollary 3.5. *If C is a connected curve in $\mathbb{P}^n$ and $\Xi(C) = 3$, then C is a curve of maximal regularity.* □

We now construct examples of connected curves of maximal regularity with no extremal secant lines. By Proposition 4.2, an example in $\mathbb{P}^4$ gives rise to examples in $\mathbb{P}^n$ for all $n \geq 4$. Therefore we shall give the construction in $\mathbb{P}^4$. In fact, for the curves we construct in $\mathbb{P}^4$ we will even show that they have no higher dimensional extremal secants as well. Before beginning the construction we prove several lemmas that will be used in the proof.

Lemma 3.6. *If $C \subset \mathbb{P}^n$ is a connected nondegenerate curve and H is a hyperplane containing no components of C, then $C \cap H$ is nondegenerate in H.*

Proof. There is a commutative diagram with exact rows

$$0 \longrightarrow \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}) \longrightarrow \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}(1)) \longrightarrow \mathrm{H}^0(\mathcal{O}_H(1)) \longrightarrow 0$$
$$ \alpha \downarrow \beta \downarrow \gamma \downarrow$$
$$0 \longrightarrow \mathrm{H}^0(\mathcal{O}_C) \longrightarrow \mathrm{H}^0(\mathcal{O}_C(1)) \longrightarrow \mathrm{H}^0(\mathcal{O}_{C \cap H}(1)).$$

C being nondegenerate is equivalent to the statement that β is injective. C being connected is equivalent to the statement that α is surjective. It follows from the snake lemma that γ is injective. This says precisely that $C \cap H$ is nondegenerate in H. □

Lemma 3.7. *If $C \subset \mathbb{P}^n$ is a connected nondegenerate curve and L is a linear subspace containing no components of C, then*

$$\deg(C \cap L) \leq \deg(C) - n + 1 + \dim(L) = \Xi(C) + \dim(L) - 1.$$

Proof. Choose a hyperplane H containing L but not containing any component of C, and consider the following commutative triangle:

$$\mathrm{H}^0(\mathcal{O}_H(1)) \xrightarrow{\alpha} \mathrm{H}^0(\mathcal{O}_{C \cap H}(1))$$
$$\beta \searrow \downarrow \gamma$$
$$\mathrm{H}^0(\mathcal{O}_{C \cap L}(1))$$

Since $C \cap H$ is nondegenerate in H, the map α is injective. Since $C \cap L \subset L$,

$$\dim_{\mathbb{K}}(\ker(\beta)) \geq \dim(H) - \dim(L).$$

Therefore $\dim_{\mathbb{K}}(\ker(\gamma)) \geq \dim(H) - \dim(L)$. Since $C \cap H$ is a finite scheme, γ is surjective. Therefore

$$\deg(C \cap L) = \dim_{\mathbb{K}}(\mathrm{H}^0(\mathcal{O}_{C \cap L}(1)))$$
$$= \dim_{\mathbb{K}}(\mathrm{H}^0(\mathcal{O}_{C \cap H}(1))) - \dim_{\mathbb{K}}(\ker(\gamma))$$
$$\leq \deg(C) - (n - 1) + \dim(L).$$

□

Lemma 3.8. *If $X \subset \mathbb{P}^2$ is a finite scheme of degree d, then $\mathrm{reg}(X) \leq d$. Moreover, if $\mathrm{reg}(X) = d$, then X lies on a line, $M \subset \mathbb{P}^2$. If $\mathrm{reg}(X) = d - 1$ and $d \neq 4$, then there exists a line $M \subset \mathbb{P}^2$ such that $\deg(M \cap X) = d - 1$.*

Proof. By Proposition 3.7 and Corollary 3.9 of [3], if

$$0 \to \sum_{i=1}^{t} S(-b_i) \xrightarrow{M} \sum_{i=1}^{t+1} S(-a_i) \to S$$

is a free resolution of S_X with $a_1 \geq ... \geq a_{t+1}$ and $b_1 \geq ... \geq b_t$, and if we let e_i and f_i denote the degrees of the entries on the principal diagonals of M, then for all i,

- $e_i \geq 1$, $f_i \geq 1$,
- $f_i \geq e_i$, $f_i \geq e_{i+1}$,
- $a_i = \sum_{j<i} e_j + \sum_{j \geq i} f_j$,
- $b_i = a_i + e_i$, for $1 \leq i \leq t$ and $\sum_{i=1}^{t} b_i = \sum_{i=1}^{t+1} a_i$,
- $d = \sum_{i \leq j} e_i f_j$.

Notice that this implies that $b_1 - 1 = a_1 + e_1 - 1 \geq a_1$. Therefore,

$$\text{reg}(X) = \max\{a_1, b_1 - 1\} = b_1 - 1 = a_1 + e_1 - 1 = \sum_{i=1}^{t} f_t + e_1 - 1.$$

Suppose that $t \geq 2$. Then

$$\sum_{i=1}^{t} f_t + e_1 - 1 \leq \sum_{i=1}^{t} e_1 f_t < \sum_{i \leq j} e_i f_j = d.$$

Suppose that $t = 1$. Then

$$\sum_{i=1}^{t} f_t + e_1 - 1 = f_1 + e_1 - 1 \leq e_1 f_1 = d.$$

Therefore $\text{reg}(X) \leq d$.

Suppose that $\text{reg}(X) = d$. By the above argument, $t = 1$ and one of e_1 or f_1 equal to 1. Since $f_1 \geq e_1$, this implies that $e_1 = 1$. Therefore $a_2 = 1$ and there is some line $M \subset \mathbb{P}^2$ such that $X \subset M$.

Suppose that $\text{reg}(X) = d - 1$ and $d \neq 4$. If $d = 3$, we are done as X has a degree 2 subscheme which must then lie on a line. Therefore we may assume that $d \geq 5$. By the above argument, either $t = 1$ and $e_1 = f_1 = 2$, or $t \geq 2$ and

$$(7) \qquad \sum_{i=1}^{t} f_t + e_1 = \sum_{i=1}^{t} e_1 f_t + 1 = \sum_{i \leq j} e_i f_j.$$

In the first case we would have $d = e_1 f_1 = 4$ which we already ruled out by assuming $d \geq 5$. Therefore we must be in the second case. In this case, equation (7) implies that $e_1 = 1$, $t = 2$, and $e_2 = f_2 = 1$. Also, we must have that $b_1 = d$ since $\text{reg}(X) = b_1 - 1$. Therefore $a_1 = b_1 - e_1 = d - 1$ and $a_2 = e_1 + f_2 = 2$. X cannot lie on a line since then it would have regularity d. Therefore $a_3 \geq 2$, so $a_3 = 2$ as $2 = a_2 \geq a_3$. Also,

$$d + b_2 = b_1 + b_2 = a_1 + a_2 + a_3 = d - 1 + 2 + 2 = d + 3$$

so $b_2 = 3$. Therefore the free resolution of S_X has the form:

$$0 \to S(-d) \oplus S(-3) \to S(-d+1) \oplus S(-2) \oplus S(-2) \to S.$$

Let f, g, and h form a generating set for I_X with f and g of degree 2 and h of degree $d - 1$. If f is a nonzerodivisor modulo g, then $Z((f, g))$ would be a complete intersection of degree 4 containing X. This is a contradiction to the assumption

that $\deg(X) \geq 5$, so f must be a zerodivisor modulo g. The only way this can happen is if $f = ml_1$ and $g = ml_2$ for some linear forms m, l_1, and l_2 with $l_1 \neq l_2$. If $h \in (m)$, then the line M defined by m would lie inside of X. Therefore h is a nonzerodivisor modulo m and $Z((h, m)) = M \cap X$ is a finite scheme of degree $d - 1$. $\square$

Lemma 3.9. *If X is a finite subscheme of $\mathbb{P}^3$ lying in a plane O with $\deg(X) \geq 2$ and Y is a finite subscheme of $\mathbb{P}^3$ of degree 2 which does not meet O, then $\mathrm{reg}(X \cup Y) \leq \mathrm{reg}(X)$.*

Proof. The result follows from the exact sequence

$$0 \to I_{X \cup Y} \to I_X \oplus I_Y \to I_X + I_Y \to 0$$

after observing that $I_X + I_Y$ is either $\mathfrak{m}$ or $\langle x_0^2, x_1, x_2, x_3 \rangle$ under a suitable choice of coordinates. $\square$

Construction. (For any $m \geq 4$ there exists a curve, C, in $\mathbb{P}^4$ of degree $m+2$ having maximal regularity, but having no extremal secant.): Let $m \geq 4$. Pick three 2-planes L, M, and N such that M and N meet L in lines, but meet each other in a single point. Pick a line K in L meeting M and N in distinct points P and Q. Pick a conic F in M which meets L in a double point whose reduced structure is contained in K and a conic G in N which meets L in a double point whose reduced structure lies in K. Pick a curve E in L of degree $m - 3$ not meeting either P or Q and not containing K. Set $C = E \cup F \cup G \cup K$. (See the introduction for an example with $m = 7$.)

The span of C is all of $\mathbb{P}^4$ so the regularity of C is at most $\Xi(C) = m - 3 + 5 - 4 + 2 = m$ as $\deg(C) = m - 3 + 5$. We will show that I_C requires a generator of degree at least m, hence has regularity at least m, and so exactly m.

To prove this it suffices to show that any form $f \in I_C$ with $\deg(f) \leq m - 1$ vanishes on the divisor $2K \subset L$. Under the map $\mathbb{K}[x_0, ..., x_4] \to \mathbb{K}[y_0, y_1, y_2]$ defining the embedding of L in $\mathbb{P}^4$, f must map to an element, which we also call f, of $I_{C \cap L/L}$. Let g be the generator of $I_{E/L}$. Then since $E \subset C \cap L$, we must have that g divides f. Let $h = f/g$ and let $D = F \cup G \cup K$. We can see that $h \in I_{D \cap L/L}$ as it still must vanish on K and f and h differ only by a unit locally at P and Q. We can also see that h is either zero or has degree at most 2. In either case it must be a multiple of k^2 where k is the linear form defining K as this is the only nonzero element of degree at most 2 in $I_{D \cap L/L}$ up to scalar multiples. Therefore f must vanish on $2K \subset L$.

We must now prove that C does not have an m-secant line. To do this, let H be any line in $\mathbb{P}^4$ not contained in C. Suppose that H does not lie in $L \cup M \cup N$. Then it can meet $L \cup M \cup N$ in a scheme of length at most 3, hence it can be at most a 3-secant to C. Since $m \geq 4$, such an H is not an extremal secant to C. Suppose that H lies in M but not in L. Then it can meet $L \cup C$ in a scheme of length at most 3, hence it can be at most a 3-secant to C. Again this implies that such an H is not an extremal secant to C. By symmetry the same applies to any line which lies in N but not L, so suppose that H lies in L. Suppose that H does not meet either P or Q. Then $H \cap C = H \cap (K \cup E)$ so H is an $(m-2)$-secant to C. Again, such an H is not an extremal secant to C, so suppose that H passes through either P or Q. It cannot pass through both P and Q since then it would be equal to K, so it passes through exactly one of them which we may assume by symmetry is P.

Since $C \cap L$ has, locally at P, the structure of a line union a double point not in that line, $H \cap C$ has degree 2 locally at P. Elsewhere, H meets C only in E. It must meet E exactly $m - 3$ times, so H is an $(m - 1)$-secant to C. Again, such an H is not an extremal secant to C, so C has no extremal secant lines.

We now show that C does not have an extremal secant 2-plane. Suppose, to the contrary, that H is such an extremal secant 2-plane. By Lemma 3.7, the degree of $C \cap H$ is at most $\Xi(C) + 1$. Since $\Xi(C) = m \geq 4$, Lemma 3.8 implies that there must be some line M in H such that $M \cap (C \cap H)$ has degree $\Xi(C)$. But such an M would be an extremal secant line to C which we have already shown does not exist. Therefore C does not have an extremal secant 2-plane.

Finally, we need to show that C does not have an extremal secant hyperplane. Suppose that C has an extremal secant hyperplane H. If H passed through both P and Q, it would contain K, which cannot happen since $H \cap C$ is finite. By symmetry we may assume it does not contain Q. In this case,

$$H \cap C = (H \cap (E \cup F \cup K)) \cup (H \cap G).$$

Applying Lemma 3.9 with $X = H \cap (E \cup F \cup K)$ and $Y = H \cap G$, we see that $H \cap \mathrm{Span}(E \cup F \cup K)$ is an extremal secant 2-plane to C. Since we have already shown that these don't exist, we have a contradiction. Therefore C has no extremal secants. $\square$

4. Curves of maximal regularity: Structure theorems

Despite the example above, it turns out that many connected curves of maximal regularity do have an extremal secant line. One way one might try to show this is by breaking a connected curve into smaller connected components and seeing how they can fit together. The following corollary, which follows from Theorem 2.1 and Lemma 2.2, says, in essence, that these components have to be fairly well separated.

Corollary 4.1. *If C and D are intersecting connected curves in $\mathbb{P}^n$ having no common components such that $C \cup D$ has maximal regularity, then $\dim(\mathrm{Span}(C) \cap \mathrm{Span}(D)) \leq 2$.* $\square$

This corollary allows us to split our analysis of connected curves of maximal regularity into three major cases depending on $\dim(\mathrm{Span}(C) \cap \mathrm{Span}(D))$ where $C \cup D$ is the curve we are analyzing. In case $\dim(\mathrm{Span}(C) \cap \mathrm{Span}(D)) = 0$, we have a complete classification which is the content of the following proposition.

Proposition 4.2. *If C and D are intersecting connected curves in $\mathbb{P}^n$ having no common components such that $\dim(\mathrm{Span}(C) \cap \mathrm{Span}(D)) = 0$, and $\Xi(C) \leq \Xi(D)$, then the curve $C \cup D$ has maximal regularity if and only if C is a tree of rational normal curves and D is either a line or has maximal regularity. Moreover, $C \cup D$ has an extremal secant line if and only if D is either a line or has an extremal secant line.*

Proof. Inspecting the proof of Proposition 2.8, if $\Xi(D) \geq 3$, then we see that there is equality throughout the chain of inequalities if and only if $\mathrm{reg}(D) = \Xi(D)$ and $\Xi(C) = 2$. If $\Xi(D) = 2$, then $\Xi(C) = 2$ and $\Xi(C \cup D) = 2$, so by Proposition 3.4 it follows that $C \cup D$ has maximal regularity and D is either a line or has maximal regularity. In either of these cases it follows from Proposition 3.4 that C is a tree. Therefore C is a tree and D is either a line or has maximal regularity.

If $\Xi(D) = 2$, then, by the above, $C \cup D$ is a tree distinct from a line hence it has an extremal secant line. Also, D is either a line or has an extremal secant line. Suppose that $\Xi(D) \geq 3$. If D has an extremal secant line, then that same line is an extremal secant for $C \cup D$ as $\Xi(D) = \Xi(C \cup D)$. Suppose that $C \cup D$ has an extremal secant line L. If L did not lie in either $\mathrm{Span}(C)$ or $\mathrm{Span}(D)$, then L can intersect $C \cup D$ in at most a scheme of length 2, so L must lie in one of $\mathrm{Span}(C)$ or $\mathrm{Span}(D)$. Notice that

$$(C \cup D) \cap \mathrm{Span}(C) = C,$$
$$(C \cup D) \cap \mathrm{Span}(D) = D,$$

so if L lay in $\mathrm{Span}(C)$ it would intersect $C \cup D$ in a scheme of length at most 2. Therefore L must lie in $\mathrm{Span}(D)$. It follows that

$$L \cap (C \cup D) = L \cap D$$

and L is an extremal secant for D. $\qquad\square$

I like to think of this proposition as saying that one can add or remove "feelers" to or from a connected curve without changing whether it is a curve of maximal regularity where, by "feelers", I mean trees whose spans intersect the span of the rest of the curve in a single point. Notice that this allows us to push the example constructed at the end of Section 3 into any $\mathbb{P}^n$, $n \geq 4$, by adding these "feelers" to it.

Next suppose $\dim(\mathrm{Span}(C) \cap \mathrm{Span}(D)) = 1$. This case splits naturally into two subcases depending on whether the line that is the intersection of the spans is contained in $C \cup D$ or not. We (partially) deal with the second case in the next proposition.

Proposition 4.3. *If C and D are intersecting connected curves in $\mathbb{P}^n$ having no common components such that $\dim(\mathrm{Span}(C) \cap \mathrm{Span}(D)) = 1$, $L = \mathrm{Span}(C) \cap \mathrm{Span}(D) \not\subseteq C \cup D$, and neither C nor D is a tree, then $C \cup D$ has maximal regularity if and only if L is an extremal secant to $C \cup D$. Moreover, in this case C and D meet in a single point and L is an extremal secant to both C and D.*

Proof. If L is an extremal secant to $C \cup D$, then $C \cup D$ is a curve of maximal regularity. So we need only prove the opposite direction.

Assume that $\mathrm{reg}(C \cup D) = \Xi(C \cup D)$. Using the notation from the first case in the proof of Proposition 2.7, we see that

$$\mathrm{reg}(C \cup D) = \Xi(C \cup D) = \Xi(C) + \Xi(D) - 1$$

implies that

$$\max\{\mathrm{reg}(C), \mathrm{reg}(D), \mathrm{reg}(I_C + I_D) + 1\} = \Xi(C) + \Xi(D) - 1.$$

Since $\mathrm{reg}(C) \leq \Xi(C)$, $\mathrm{reg}(D) \leq \Xi(D)$, and both $\Xi(C)$ and $\Xi(D)$ are greater than or equal to 2 this implies that

$$\mathrm{reg}(I_C + I_D) = \Xi(C) + \Xi(D) - 2.$$

Equation (5) implies that

$$\mathrm{reg}(I_C + I_D) \leq \max\{\deg((C \cap L) \cup (D \cap L)) - 1, \Xi(C), \Xi(D)\}.$$

Since neither C nor D is a tree, Proposition 3.4 implies that $\Xi(C)$ and $\Xi(D)$ are both at least 3, so

$$(8) \qquad \deg((C \cap L) \cup (D \cap L)) = \Xi(C) + \Xi(D) - 1 = \Xi(C \cup D).$$

It follows that L is an extremal secant to $C \cup D$ as

$$(C \cap L) \cup (D \cap L) \subset (C \cup D) \cap L.$$

Furthermore, since, by Lemma 2.4,

$$\deg((C \cap L) \cup (D \cap L)) = \deg(C \cap L) + \deg(D \cap L) - \deg(C \cap D)$$

it follows from equation (8) that $\deg(C \cap L) = \Xi(C)$, $\deg(D \cap L) = \Xi(D)$, and $\deg(C \cap D) = 1$. Therefore C and D meet in a single point and L is an extremal secant to both C and D. $\square$

In one sense this proposition is more satisfactory than Proposition 4.2, and in another it is less so. It is more satisfactory in the sense that if you have a curve that splits in this way, then you immediately deduce that the curve has an extremal secant without analyzing the subcurves any further. It is less satisfactory because of the "neither C nor D is a tree" condition. However, this condition cannot be removed. If we break up the curve C constructed at the end of the previous section as $F \cup (G \cup E \cup K)$, then the intersection of the spans is a line not contained in C even though C doesn't have an extremal secant line.

In fact, given any nonplanar connected curve D with an extremal secant and any tree C, one can put these together in such a way that the intersection of their spans is a line which is not an extremal secant to $C \cup D$ even though $C \cup D$ has extremal secant lines. The following example demonstrates this.

Construction. Let N be a 3-plane in $\mathbb{P}^4$ and let D be a twisted cubic in N. Let L be a secant to D and let M be a line which intersects D and L, but which is not a secant to D. (This is actually true for every line other than L that passes through D and L and does so in distinct points.) Let C be a conic not contained in N whose span contains M and which intersects both D and L. A picture of such a configuration of curves follows:
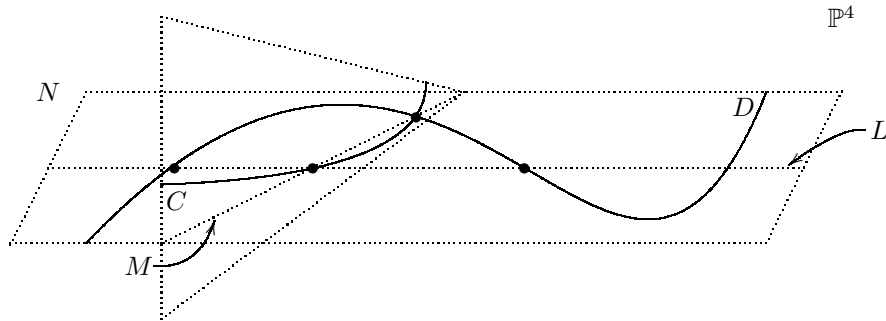
L is a 3-secant line to $C \cup D$ and $\Xi(C \cup D) = \Xi(C) + \Xi(D) - 1 = 3$ so L is an extremal secant to $C \cup D$. However by applying Lemma 3.7 to the 2-plane spanned by L and M, which is the intersection of the spans of C and D, we see that M is only a 2-secant to $C \cup D$, hence not an extremal secant. $\square$

The next proposition deals, (again, only partially), with the case when

$$\dim(\text{Span}(C) \cap \text{Span}(D)) = 1 \quad \text{and} \quad \text{Span}(C) \cap \text{Span}(D) \subset C \cup D.$$

Proposition 4.4. *If C and D are intersecting connected curves in $\mathbb{P}^n$ having no common components such that $\dim(\text{Span}(C) \cap \text{Span}(D)) = 1$ and $L = \text{Span}(C) \cap \text{Span}(D) \subset C$, then $C \cup D$ has maximal regularity if and only if C is a tree and $L \cup D$ has maximal regularity. Moreover, if $L \cup D$ has maximal regularity, then so does D.*

Proof. First of all, suppose that C is a tree and $L \cup D$ has maximal regularity. We can write $C = L \cup \bigcup_{i=1}^{k} C_i$ where the C_i's are the connected components of $\overline{C \setminus L}$. By Proposition 3.4, all the C_i's are trees and

$$\dim(\text{Span}(L \cup \bigcup_{i=1}^{j} C_i) \cap \text{Span}(C_{j+1})) = 0, \quad 0 \le j \le k - 1.$$

Since $\text{Span}(C) \cap \text{Span}(D) = L$, it follows that

$$\dim(\text{Span}(D \cup L \cup \bigcup_{i=1}^{j} C_i) \cap \text{Span}(C_{j+1})) = 0, \quad 0 \le j \le k - 1.$$

By Proposition 4.2 and induction on j, it follows that $D \cup L \cup \bigcup_{i=1}^{j} C_i$ has maximal regularity for all j, $0 \le j \le k$. In particular, $C \cup D$ has maximal regularity.

Now assume that $C \cup D$ has maximal regularity. From the proof of case b of Proposition 2.7, it follows that

$$\max\{\Xi(C), \Xi(D) + 1\} = \Xi(C) + \Xi(D) - 1.$$

Since $\Xi(D) \ge 2$, this implies that $\Xi(C) = 2$. By Proposition 3.4 this implies that C is a tree.

To see that $L \cup D$ has maximal regularity, we let $C = L \cup \bigcup_{i=1}^{k} C_i$ where the C_i's are the connected components of $\overline{C \setminus L}$. As in the first part of this proof, all the C_i's are trees and

$$\dim(\text{Span}(D \cup L \cup \bigcup_{i=1}^{j} C_i) \cap \text{Span}(C_{j+1})) = 0, \quad 0 \le j \le k - 1.$$

Therefore, by Proposition 4.2 and reverse induction on j, $D \cup L \cup \bigcup_{i=1}^{j} C_i$ has maximal regularity for all j, $0 \le j \le k$. In particular, $L \cup D$ has maximal regularity.

Now suppose that $L \cup D$ has maximal regularity. By Theorem 2.1, the regularity of $L \cup D$ is at most 1 more than the regularity of D. Since $\Xi(L \cup D) = \Xi(D) + 1$, it follows that D must have maximal regularity. $\square$

To finish the analysis we would need to answer the question, "When can one add a line to a curve of maximal regularity such that the resulting curve has maximal regularity?" It might be tempting to guess that you have to add this line in such a way that it passes through an extremal secant line in a point that the original curve did not as this will always cause the resulting curve to have maximal regularity. This is not the case as one can see from the construction in Section 3 by choosing F to be the union of two lines Y and Z and splitting C up as $Y \cup (Z \cup G \cup E \cup K)$.

The last case is $\dim(\text{Span}(C) \cap \text{Span}(D)) = 2$. We break this case into subcases depending on whether $C \cup D$ has components in $\text{Span}(C) \cap \text{Span}(D)$ or not. We first deal with the case when no component of $C \cup D$ lies in $\text{Span}(C) \cap \text{Span}(D)$.

Like Proposition 4.3, it turns out that such a curve of maximal regularity must have an extremal secant without looking any further at subcurves. In this case we don't have the restriction that neither of the components be trees.

Proposition 4.5. *If C and D are intersecting connected curves in $\mathbb{P}^n$ having no common components such that $L = \mathrm{Span}(C) \cap \mathrm{Span}(D)$ has dimension 2 and no component of either C or D lies in L, then $C \cup D$ has maximal regularity if and only if there is some line $M \subset L$ which is an extremal secant for $C \cup D$.*

Proof. If $C \cup D$ has an extremal secant lying in L, then it has maximal regularity, so we need only prove the reverse direction.

Assume that $\mathrm{reg}(C \cup D) = \Xi(C \cup D)$. Then from the exact sequence

$$0 \rightarrow I_{C \cup D} \rightarrow I_C \oplus I_D \rightarrow I_C + I_D \rightarrow 0$$

we see that $\mathrm{reg}(C \cup D) \leq \max\{\mathrm{reg}(C), \mathrm{reg}(D), \mathrm{reg}(I_C + I_D) + 1\}$. By the Main Theorem we have that $\mathrm{reg}(C) \leq \Xi(C)$ and $\mathrm{reg}(D) \leq \Xi(D)$, so from Lemma 2.2 we see that $\mathrm{reg}(I_C + I_D) \geq \Xi(C \cup D) - 1$. By Theorem 2.1,

$$\mathrm{reg}(I_C + I_D) \leq \mathrm{reg}(C) + \mathrm{reg}(D) - 1 \leq \Xi(C \cup D) - 1.$$

Therefore

(9) $$\mathrm{reg}(I_C + I_D) = \Xi(C \cup D) - 1.$$

Now let $J_1 = I_C + I_L$ and $J_2 = I_D + I_L$. Since $I_L \subset I_C + I_D$, we see that $J_1 + J_2 = I_C + I_D$. From the exact sequence

$$0 \rightarrow J_1 \cap J_2 \rightarrow J_1 \oplus J_2 \rightarrow J_1 + J_2 = I_C + I_D \rightarrow 0$$

it follows that

$$\mathrm{reg}(I_C + I_D) \leq \max\{\mathrm{reg}(J_1), \mathrm{reg}(J_2), \mathrm{reg}(J_1 \cap J_2) - 1\}.$$

By Theorem 2.1 and the Main Theorem,

(10) $$\mathrm{reg}(J_1) \leq \mathrm{reg}(C) \leq \Xi(C),$$

(11) $$\mathrm{reg}(J_2) \leq \mathrm{reg}(D) \leq \Xi(D).$$

Therefore $\mathrm{reg}(I_C + I_D) \leq \max\{\Xi(C), \Xi(D), \mathrm{reg}(J_1 \cap J_2) - 1\}$. By equation (9), this implies that $\mathrm{reg}(J_1 \cap J_2) \geq \Xi(C \cup D)$. By Theorem 2.1 this inequality cannot be strict, so

(12) $$\mathrm{reg}(J_1 \cap J_2) = \Xi(C \cup D).$$

By equation (3), $\mathrm{reg}(J_1 \cap J_2) = \max\{\mathrm{reg}((J_1 \cap J_2)^{sat}), \mathrm{sat}(J_1 \cap J_2)\}$. From this, inequalities (4), (10), and (11), and equation (12) we see that

$$\mathrm{reg}((J_1 \cap J_2)^{sat}) = \Xi(C \cup D).$$

The ideal of $(C \cap L) \cup (D \cap L)$ is $(J_1 \cap J_2)^{sat}$. Since $(C \cap L) \cup (D \cap L)$ lies in $(C \cup D) \cap L$, $\deg((C \cap L) \cup (D \cap L)) \leq \deg((C \cup D) \cap L)$. By Lemma 3.7,

$$\deg((C \cup D) \cap L) \leq \Xi(C \cup D) + 1.$$

Therefore $(C \cap L) \cup (D \cap L)$ is a finite subscheme of a 2-plane, L, of degree at most $\Xi(C \cup D) + 1$ and regularity exactly $\Xi(C \cup D)$. Furthermore,

$$\Xi(C \cup D) = \Xi(C) + \Xi(D) \geq 4,$$

so either the degree of $(C \cap L) \cup (D \cap L)$ is exactly equal to its regularity, or it is at least 5 and the difference between them is 1. In either case, Lemma 3.8 implies that there is a line $M \subset L$ such that

$$\deg(M \cap ((C \cap L) \cup (D \cap L))) = \Xi(C \cup D).$$

This implies that the length of $M \cap (C \cup D)$ is at least $\Xi(C \cup D)$, hence M is an extremal secant line to $C \cup D$. $\square$

We now turn our attention to the case when $\dim(\mathrm{Span}(C) \cap \mathrm{Span}(D)) = 2$ and there are components of $C \cup D$ that lie in $\mathrm{Span}(C) \cap \mathrm{Span}(D)$. We need two lemmas about subschemes of $\mathbb{P}^2$.

Lemma 4.6. *If $X = D \cup Y$ is a subscheme of $\mathbb{P}^2$ where D is a curve of degree d and Y is a finite scheme, then*

$$0 \to \sum_{i=1}^{t} S(-b_i) \xrightarrow{M} \sum_{i=1}^{t+1} S(-a_i) \to S$$

is a free resolution of S_X with $a_1 \geq \ldots \geq a_{t+1}$ and $b_1 \geq \ldots \geq b_t$, and e_i and f_i denote the degrees of the entries on the principal diagonals of M, then for all i,

- $e_i \geq 1$, $f_i \geq 1$,
- $f_i \geq e_i$, $f_i \geq e_{i+1}$,
- $a_i = \sum_{j<i} e_j + \sum_{j\geq i} f_j + d$,
- $b_i = a_i + e_i$, for $1 \leq i \leq t$ and $\sum_{i=1}^{t} b_i + d = \sum_{i=1}^{t+1} a_i$,
- $H_{S_X}(n) = dn + 1 - (d-1)(d-2)/2 + \sum_{i\leq j} e_i f_j$.

The proof of this lemma is analogous to Proposition 3.7 and Corollary 3.9 of [3] and we do not repeat it here. The main difference is that the generators of the ideal are not the maximal minors of M but rather the maximal minors of M multiplied by the equation of D which is where you get all the "$+d$" terms above as well as the difference in the Hilbert polynomial.

Lemma 4.7. *If $\emptyset \neq D \subset \mathbb{P}^2$ is a curve of degree d and $Y \subset \mathbb{P}^2$ is a finite scheme, then $\mathrm{reg}(D \cup Y) \leq d + \deg(Y) - \deg(D \cap Y)$. Moreover, if we have equality and $\deg(D \cap Y) = 1$, then there is some line, $M \subset \mathbb{P}^2$, such that*

$$\deg(M \cap (D \cup Y)) = d + \deg(Y) - 1.$$

Proof. If $Y \subset D$, then everything is trivial, so we may assume that $Y \not\subset D$. From the exact sequence

$$0 \to S_{D \cup Y} \to S_D \oplus S_Y \to S/(I_D + I_Y) \to 0$$

and the fact that $S/(I_D + I_Y)$ agrees with $S_{D \cap Y}$ in high degrees we see that

$$H_{S_{D \cup Y}}(n) = H_{S_D}(n) + H_{S_Y}(n) - H_{S_{D \cap Y}}(n)$$
$$= dn + 1 - (d-1)(d-2)/2 + \deg(Y) - \deg(D \cap Y).$$

Therefore, using the notation from Lemma 4.6 applied to $D \cup Y$,

$$\sum_{i\leq j} e_i f_j = \deg(Y) - \deg(D \cap Y).$$

Also,

$$\text{reg}(D \cup Y) = \max\{a_1, b_1 - 1\} = b_1 - 1$$

$$= a_1 + e_1 - 1 = \sum_{i=1}^{t} f_t + e_1 + d - 1.$$

If $t \geq 2$, then

$$\sum_{i=1}^{t} f_t + e_1 + d - 1 \leq \sum_{i=1}^{t} e_1 f_t + d < \sum_{i \leq j} e_i f_j + d = d + \deg(Y) - \deg(D \cap Y).$$

If $t = 1$, then

$$\sum_{i=1}^{t} f_t + e_1 + d - 1 = f_1 + e_1 + d - 1 \leq e_1 f_1 + d = d + \deg(Y) - \deg(D \cap Y).$$

Therefore $\text{reg}(D \cup Y) \leq d + \deg(Y) - \deg(D \cap Y)$.

Suppose that $\deg(D \cap Y) = 1$ and $\text{reg}(D \cup Y) = d + \deg(Y) - 1$. By the above argument, this implies that $t = 1$ and one of e_1 or f_1 equal to 1. Since $f_1 \geq e_1$, this implies that $e_1 = 1$. Therefore $a_2 = d + e_1 = d + 1$. Also, we must have that $b_1 = d + \deg(Y)$, so the free resolution of $S_{D \cup Y}$ is

$$0 \rightarrow S(-(d + \deg(Y))) \rightarrow S(-(d + \deg(Y)) + 1) \oplus S(-(d+1)) \rightarrow S.$$

Let g be the equation of D. Then the above says that the minimal generators of $I_{D \cup Y}$ have the form gm and gh where m is some linear form and h is a form of degree $\deg(Y) - 1$. Moreover, m does not divide h as otherwise gh wouldn't be a minimal generator. Therefore, if m does not divide g, then $Z((gh, m)) = M \cap (D \cup Y)$ is a finite scheme of degree $d + \deg(Y) - 1$ where M denotes the line defined by m.

Suppose that m divides g. Since $\deg(D \cap Y) = 1$ and $Y \nsubseteq D$, it follows that Y must be a double point not lying in D whose reduced point is a nonsingular point of D lying on M. In this case, h is a linear form which does not divide g. Therefore, as above, h defines a line which intersects $D \cup Y$ in a finite scheme of length $d + 1$. $\qquad\square$

Proposition 4.8. *If C and D are intersecting connected curves in $\mathbb{P}^n$ having no common components such that $L = \text{Span}(C) \cap \text{Span}(D)$ has dimension 2, no component of C lies in L, and $D \subset L$, then $C \cup D$ has maximal regularity if and only if there is some line $M \subset L$ which is an extremal secant for $C \cup D$.*

Proof. If $C \cup D$ has an extremal secant lying in L, then it has maximal regularity, so we need only prove the reverse direction.

Assume that $\text{reg}(C \cup D) = \Xi(C \cup D)$. Then exactly as in the proof of Proposition 4.5, we have that $\text{reg}(I_C + I_D) = \Xi(C \cup D) - 1$. Let $J = I_C + I_L$. Since $I_L \subset I_C + I_D$, we see that $J + I_D = I_C + I_D$. Again, as in the proof of Proposition 4.5, from the exact sequence

$$0 \rightarrow J \cap I_D \rightarrow J \oplus I_D \rightarrow J + I_D = I_C + I_D \rightarrow 0$$

we deduce that $\text{reg}((J \cap I_D)^{sat}) = \Xi(C \cup D)$. Since $Z((J \cap I_D)^{sat}) = D \cup (C \cap L)$, Lemma 4.7 implies that

$$(13) \qquad \deg(C \cap D) \leq \deg(D) + \deg(C \cap L) - \Xi(C \cup D).$$

By Lemma 3.7,

$$(14) \qquad \deg(C \cap L) \leq \Xi(C) + 1.$$

Also, $\Xi(D) = \deg(D)$ and $\Xi(C \cup D) = \Xi(C) + \Xi(D)$. Therefore

$$\deg(C \cap D) = 1.$$

Moreover, we must have equality in (13) and (14) and thus by Lemma 4.7, there is a line $M \subset L$, such that $\deg(M \cap (D \cup (C \cap L))) = \Xi(C \cup D)$. Since

$$D \cup (C \cap L) \subset (C \cup D) \cap L,$$

M is an extremal secant to $C \cup D$. $\square$

These results allow us to prove that many of the curves of maximal regularity in $\mathbb{P}^3$ have extremal secants.

Theorem 4.9. *If $C \subset \mathbb{P}^3$ is a connected curve with no linear components, then C has maximal regularity if and only if either $\Xi(C) = 3$ or C has an extremal secant.*

Proof. We have already proved the "if" part of this theorem, so assume that C has maximal regularity. If C is planar, then C has an extremal secant.

Assume that C is nondegenerate in $\mathbb{P}^3$. If C is irreducible, then we are done by Theorems 2.1 and 3.1 of [6].

Assume that C is reducible. We can find two connected curves with no common components, D and E, such that $C = D \cup E$. In fact, if A and B are planar components of C lying in the same plane, then we can (and do) ensure that A and B either both lie in D or both lie in E. If D is nondegenerate in $\mathbb{P}^3$ and E is planar, then Proposition 4.8 implies that C has an extremal secant in the plane that E spans.

Assume that D and E are both planar and the intersection of their spans, L, is a line lying in neither of them. If neither D nor E is a conic, then Proposition 4.3 implies that L is an extremal secant to C.

Assume that D is a conic. If $\deg(D \cap E) = 1$, then

$$\deg((D \cap L) \cup (E \cap L)) = \deg(D) + \deg(E) - 1 = \Xi(C).$$

Since $(D \cap L) \cup (E \cap L) \subset C \cap L$ this implies that L is an extremal secant to C.

Assume that $\deg(D \cap E) \geq 2$. Since D is a conic, $\deg(D \cap L) = 2$, so $\deg(D \cap E) = 2$. Notice that $I_D + I_E$ contains I_L. It also contains a polynomial, f, of degree 2 which defines D in its span. However, $I_L + (f)$ is the (saturated) homogeneous ideal of a scheme of length 2 on L. Therefore, since $D \cap E$ is a subscheme of this scheme and also has length 2, we must have equality and $I_D + I_E = I_{D \cap E}$. In particular, since D, E, and $D \cap E$ are all arithmetically Cohen-Macaulay, C is arithmetically Cohen-Macaulay. We now let H be any plane such that $Z = C \cap H$ is finite. Since C is arithmetically Cohen-Macaulay, $\mathrm{reg}(Z) = \mathrm{reg}(C)$. If $\deg(C) = 4$, then $\Xi(C) = 3$ and we are done.

Assume that $\deg(C) \geq 5$. Then $\deg(Z) \geq 5$, and Proposition 3.8 implies that there is some line M such that $\deg(M \cap Z) = \Xi(C)$. In particular, $\deg(M \cap C) = \Xi(C)$ and M is an extremal secant to C. This finishes the proof. $\square$

Finally, we mention the following corollaries of the analysis in this section.

Corollary 4.10. *If C and D are intersecting connected curves in $\mathbb{P}^n$ having no common components and $C \cup D$ has maximal regularity, then C and D are both either lines or curves having maximal regularity.*

Corollary 4.11. *Any connected subcurve of a curve of maximal regularity is either a line or has maximal regularity.*

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